Effective action and decoherence by fermions in quantum cosmology

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Abstract

We develop the formalism for the one-loop no-boundary state in a cosmological model with fermions. We use it to calculate the reduced density matrix for an inflaton field by tracing out the fermionic degrees of freedom, yielding both the fermionic effective action and the standard decoherence factor. We show that dimensional regularisation of ultraviolet divergences would lead to an inconsistent density matrix. Suppression of these divergences to zero is instead performed through a nonlocal Bogoliubov transformation of the fermionic variables, which leads to a consistent density matrix. The resulting degree of decoherence is less than in the case of bosonic fields.

1.   Introduction

Quantum cosmology – the application of quantum theory to the Universe as a whole – is undergoing now a period of successful development connected with the fact that it can help to explain the origin of inflation in the early Universe. One of the basic problems in quantum cosmology is the choice of an appropriate boundary condition. Two proposals have become especially popular: The no-boundary state [1, 2] and the tunneling state [3]. Analysis of these two proposals in the one-loop approximation allows one to formulate the conditions for the normalisability of the wave function of the Universe [4] and to calculate the probability peak for the energy scale of inflation in different models [5, 6]. Such a probability peak has been interpreted as providing a criterium to select amongst the members of an "ensemble" of classical universes.

However, quantum theory does not yield a classical ensemble. A complete analysis would thus have to include a quantitative discussion of the quantum-to-classical transition.

It is now generally accepted that the classical properties for a subsystem arise from the irreversible interaction of this quantum system with its natural environment. Starting with the pioneering work of Zeh in the seventies [7], this concept of *decoherence* has been developed extensively, see [8] and [9] for reviews. Quite recently, this continuous loss of coherence was observed in quantum-optical experiments [10].

Quantum-to-classical transition through decoherence also applies to quantum cosmology where it was suggested to consider background degrees of freedom (such as the scale factor and the homogeneous mode of the inflaton scalar field) as the relevant system and the perturbations (such a gravitational waves or density fluctuations) as an environment [11]. A quantitative discussion of this idea was first done in [12], where it was demonstrated how the scale factor and the inflaton field can acquire classical properties. This idea was further pursued in many papers [8]. The quantum-to-classical transition through decoherence plays also a crucial role for the primordial fluctuations itself that eventually serve as classical seeds for galaxies and clusters of galaxies [13].

However, there is one big problem arising in the application of this idea to cosmology. Since there are infinitely many environmental modes, there arise ultraviolet divergences that have to be removed. It was already noted in [14] that dimensional regularisation is only applicable for the phase part of decoherence factor, not its absolute value. In our preceding paper [15] this was confirmed by detailed calculations. It was also shown in [15], that for the case of bosonic fields, using the so-called conformal parametrisation one can get rid of ultraviolet divergences and obtain finite results for the absolute value of the decoherence factor. An analogous idea was investigated in QED and in a particular cosmological model in [16]. A similar proposal was also discussed in [17].

The purpose of the present paper is to extend the analysis of [15] to fermionic degrees of freedom. From a physical point of view, this is important because

fermions constitute the main part of physical fields in our Universe. On the other hand, the Grassmannian character of fermion variables implies an essential modification of the formalism which is of interest by itself.

First of all, we want to emphasise that the formalism of the Hartle-Hawking wave function of the Universe via a path integral *in one-loop approximation* was previously developed only for bosonic fields [4, 5, 6]. Here, we extend this formalism to fermionic fields. In this case the structure of the norm of partial wave functions, describing perturbations of different harmonics of fermionic field - an important ingredient of the one-loop formalism - is more complicated than in the case of bosonic fields where it can be reduced to calculating the Wronskian of the corresponding basis functions. Using this formalism, we can then calculate the decoherence factor for the reduced density matrix of scale factor and inflaton field. However, like in the bosonic case, ultraviolet divergences appear, as was already noted in an early paper on decoherence due to fermions [18].

By handling the divergent series within dimensional regularisation [19, 20], we obtain a renormalised expression for the absolute value of the decoherent factor. Unfortunately, however, it shows a wrong behaviour violating the consistency conditions for a density matrix as a bounded operator, let alone its decoherence properties. A similar phenomenon was observed also in the case of bosonic fields [15]. However, a simple reparametrisation of field variables by overall multiplication with an appropriate power of the scale factor allowed us there to obtain finite expressions. This does not work in the case of fermionic fields, where such a reparametrisation is incompatible with the structure of the measure on the corresponding Hilbert space of states. However, there exists another possibility: Performing a Bogoliubov transformation of field variables one can get again finite expressions for the reduced density matrix. It is shown that such Bogoliubov transformations are in a certain sense analogous to the well-known Foldy-Wouthuysen transformation of spinor variables [21]. Thus, as in the case of bosonic fields, the definition of the environment is crucial for the consistency and decoherence properties of the reduced density matrix, but in the fermionic case this definition is much more involved.

Our paper is organised as follows. In section 2 we briefly review the spinor formalism for quantum cosmology. This section includes the relation between the two- and four-component spinor formalism and presents the expansion of Weyl spinors in three-dimensional spinor harmonics with a special emphasis on the dimensionalities of the $O(d)$ spinor representation on d-dimensional spheres (necessary for dimensional regularisation). Section 3 is devoted to the quantisation of fermion fields in the Hamiltonian formalism and to the construction of the cosmological wave function. In section 4 we discuss the connection between path integral, Euclidean effective action and density matrix of fermions. Section 5 contains the presentation of contributions of massless and massive fermions into the amplitude of the decoherence factor in the reduced density matrix. In section 6 we demonstrate that the application of dimensional regularisation violates basic properties of the density matrix. Section 7 comprises the discussion of Bogoliubov

transformation of fermionic variables, allowing to get rid of ultraviolet divergences in the decoherence factor. We also exhibit the connection with the traditional Foldy-Wouthuysen transformations. Section 8 contains a brief summary of the obtained results.

2. Spinor formalism for quantum cosmology

In the main part of this paper we shall use a two-component spinor formalism as it is usually done in papers with applications to quantum cosmology [22, 23, 24]. However, at the beginning we would like to present some formulae connecting this formalism with that of four-dimensional Dirac spinors (see, for example Ref. [25]).

We shall use two-component Weyl spinors in Van der Waerden notation. There are two types of spinors with primed and unprimed indices which are transformed under the action of matrix $M \in SL(2,C)$ according to the following rules:

$$
\begin{aligned}
\psi'_A &= M_A^B \psi_B, \\
\psi'^A &= (M^{-1})_B^A \psi^B, \\
\bar{\psi}'_{\dot{A}} &= M_{\dot{A}}^{*\dot{B}} \bar{\psi}_{\dot{B}} \\
\bar{\psi}'^{\dot{A}} &= (M^{-1})_{\dot{B}}^{*\dot{A}} \bar{\psi}^{\dot{B}}.
\end{aligned}
\tag{2.1}
$$

Those with dotted indices transform under the $\left(0, \frac{1}{2}\right)$ representation of the Lorentz group, while those with undotted indices transform under the $\left(\frac{1}{2}, 0\right)$ conjugate representation. For the Minkowski metric we use the signature

$$
\eta_{mn} = \operatorname{diag}(-1, 1, 1, 1).
\tag{2.2}
$$

The Pauli matrices $\sigma_{A\dot{A}}^m$ have the following form:

$$
\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\tag{2.3}
$$

Indices can be lowered and raised by tensors ε_{AB} and $\varepsilon_{\dot{A}\dot{B}}$ with components

$$
\begin{aligned}
&\varepsilon_{11} = \varepsilon_{22} = \varepsilon^{11} = \varepsilon^{22} = 0, \\
&\varepsilon_{21} = \varepsilon^{12} = 1, \ \varepsilon_{12} = \varepsilon^{21} = -1.
\end{aligned}
\tag{2.4}
$$

(The components of ε with dotted indices are the same). It is convenient also to define

$$
\bar{\sigma}^{m\dot{A}A} = \varepsilon^{\dot{A}\dot{B}} \varepsilon^{AB} \sigma_{B\dot{B}}^m.
\tag{2.5}
$$

The anticommutation relations for the Pauli matrices are:

$$
\begin{aligned}
(\sigma^m \bar{\sigma}^n + \sigma^n \bar{\sigma}^m)_A^B &= -2\eta^{mn} \delta_A^B, \\
(\bar{\sigma}^m \sigma^n + \bar{\sigma}^n \sigma^m)_{\dot{B}}^{\dot{A}} &= -2\eta^{mn} \delta_{\dot{B}}^{\dot{A}}.
\end{aligned}
\tag{2.6}
$$

3

For four-dimensional γ - matrices and Dirac spinors we shall use the Weyl basis. In this basis, γ - matrices are given

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}, \tag{2.7}$$

while the Dirac spinor is related to the two-component Weyl spinors by

$$\Psi = \begin{pmatrix} \chi_A \\ \bar{\phi}^{\dot{A}} \end{pmatrix}. \tag{2.8}$$

It is worth noting that the canonical basis for the Dirac γ - matrices,

$$\gamma_C^0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ \gamma_C^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \tag{2.9}$$

is related to the Weyl basis (2.7) by the following similarity transformation:

$$\Gamma_W = X\Gamma_C X^{-1}, \ X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{2.10}$$

Note that the Weyl basis is more consistent from the group-theoretical point of view because in this basis the Dirac spinor is represented as a direct sum of two spinors belonging to irreducible representations of the Lorentz group (or to be precise, its covering group $SL(2, C)$). This basis is also more convenient for the description of the massless limit for ultrarelativistic fermions. At the same time, the canonical basis is convenient for studying the non-relativistic limit and for the transition from the Dirac equation to the Pauli equation. We shall come back to this question when considering the relation between Bogoliubov transformations and the Foldy-Wouthuysen representation for Dirac spinors.

Now, starting with the traditional action for Dirac spinors in four-dimensional spacetime [26, 21] and using the above relations (2.7),(2.8), one can get the following action [22]:

$$S = -\frac{i}{2} \int d^4x e(\bar{\phi}^{\dot{A}} e_{A\dot{A}}^\mu D_\mu \phi^A + \bar{\chi}^{\dot{A}} e_{A\dot{A}}^\mu D_\mu \chi^A) + h.c. - m \int d^4x e(\chi_A \phi^A + \bar{\phi}^{\dot{A}} \bar{\chi}_{\dot{A}}), \tag{2.11}$$

where D_μ is a spinorial covariant derivative, $e_{A\dot{A}}^\mu$ is the Pauli matrix contracted with the tetrad e_m^μ, and e is the determinant of the tetrad.

Varying the action (2.11), one can get the Dirac equations

$$e_{A\dot{A}}^\mu D_\mu \phi^A = im\bar{\chi}_{\dot{A}},$$
$$e_{A\dot{A}}^\mu D_\mu \chi^A = im\bar{\phi}_{\dot{A}},$$
$$e_{A\dot{A}}^\mu D_\mu \bar{\phi}^{\dot{A}} = -im\bar{\chi}_A,$$
$$e_{A\dot{A}}^\mu D_\mu \bar{\chi}^{\dot{A}} = -im\bar{\phi}_A. \tag{2.12}$$

4

In what follows we shall consider fermionic perturbations on the background of a spatially closed Friedmann model with S^3 spatial sections. We shall work within the inflationary model with the spacetime metric [2]

$$ds^2 = -dt^2 + a^2(t)d\Omega_3^2, \tag{2.13}$$

($d\Omega_3^2$ is the metric on the unit three-sphere) with a cosmological scale factor evolving according to the law

$$a(t) = \frac{1}{H}\cosh Ht, \tag{2.14}$$

where $H = H(\varphi)$ is an effective Hubble constant possibly generated by an inflaton scalar field φ.

Then one should expand the fermionic field using the full set of spinor harmonics on the three-sphere [22]:

$$\phi_A = \frac{a^{-3/2}}{2\pi}\sum_{np}\sum_q \alpha_n^{pq}[m_{np}(t)\rho_A^{nq}(x) + \bar{r}_{np}(t)\bar{\sigma}_A^{nq}(x)],$$

$$\bar{\phi}_{\dot{A}} = \frac{a^{3/2}}{2\pi}\sum_{np}\sum_q \alpha_n^{pq}[\bar{m}_{np}(t)\bar{\rho}_{\dot{A}}^{nq}(x) + r_{np}(t)\sigma_{\dot{A}}^{nq}(x)],$$

$$\chi_A = \frac{a^{-3/2}}{2\pi}\sum_{np}\sum_q \beta_n^{pq}[s_{np}(t)\rho_A^{nq}(x) + \bar{t}_{np}(t)\bar{\sigma}_A^{nq}(x)],$$

$$\bar{\chi}_{\dot{A}} = \frac{a^{3/2}}{2\pi}\sum_{np}\sum_q \beta_n^{pq}[\bar{s}_{np}(t)\bar{\rho}_{\dot{A}}^{nq}(x) + t_{np}(t)\sigma_{\dot{A}}^{nq}(x)]. \tag{2.15}$$

Here, the weight factor $a^{-3/2}$ is especially chosen in order to write the action (2.11) in a special parametrisation reflecting the conformal properties of the spinor field. It is conformally invariant in the massless case and, therefore, decouples in this parametrisation from the gravitational background. Summation over n runs from 1 to ∞, while the indices p and q run from 1 to $n(n+1)$ - the dimensionality of the irreducible representation for Weyl spinors. The time-dependent coefficients $m_{np}, r_{np}, t_{np}, s_{np}$ and their complex conjugates are taken to be odd elements of a Grassmann algebra. The constant coefficients are included for convenience in order to avoid couplings between different values of p in the expansion of the action. They may be regarded as block-diagonal matrices of dimension $n(n+1)$ whose structure can be found in [22]. The harmonics $\rho, \sigma, \bar{\rho}, \bar{\sigma}$ are normalised eigenfunctions of the Dirac operator on the three-sphere with eigenvalues $\pm(n + 1/2), n \geq 1$. Here we are using a notation that differs from [22] by a unit shift of the quantum number n. Inserting (2.15) into the action (2.11) and using the properties of spinor harmonics, one finds that the action reads

$$S = \sum_{np}\int dt \left(\frac{i}{2}(\bar{m}_{np}\dot{m}_{np} + m_{np}\dot{\bar{m}}_{np} + \bar{s}_{np}\dot{s}_{np} + s_{np}\dot{\bar{s}}_{np}\right.$$

$$\left. + \bar{t}_{np}\dot{t}_{np} + t_{np}\dot{\bar{t}}_{np} + \bar{r}_{np}\dot{r}_{np} + r_{np}\dot{\bar{r}}_{np})\right.$$

$$+\frac{1}{a}(n+\frac{1}{2})(\bar{m}_{np}m_{np}+\bar{s}_{np}s_{np}+\bar{t}_{np}t_{np}+\bar{r}_{np}r_{np})$$

$$-m(r_{np}t_{np}+\bar{t}_{np}\bar{r}_{np}+s_{np}m_{np}+\bar{m}_{np}\bar{s}_{np}) \Big)$$

$$+\frac{i}{2}\sum_{np}(\bar{m}_{np}m_{np}+\bar{s}_{np}s_{np}+\bar{t}_{np}t_{np}+\bar{r}_{np}r_{np})_{S_I}$$

$$+\frac{i}{2}\sum_{np}(\bar{m}_{np}m_{np}+\bar{s}_{np}s_{np}+\bar{t}_{np}t_{np}+\bar{r}_{np}r_{np})_{S_F}. \tag{2.16}$$

The last two lines in this equation represent the boundary terms at the initial (S_I) and final (S_F) hypersurfaces of the spacetime domain. They have to be necessarily included in order to guarantee the consistency of the variational problem for this action with *fixed unbarred variables* at S_F and *fixed barred variables* at S_I – the choice of boundary conditions characteristic for the definition of the fermionic wave function in the holomorphic representation [22]. Thus the full action has the form of a sum of fermionic actions

$$S_n = \int dt \left(\frac{i}{2}(\bar{x}\dot{x}+x\dot{\bar{x}}+\bar{y}\dot{y}+y\dot{\bar{y}})+e^{-\alpha}(n+\frac{1}{2})(\bar{x}x+\bar{y}y)-m(yx+\bar{x}\bar{y})\right)$$

$$+\frac{i}{2}(\bar{x}x+\bar{y}y)_{S_F}+\frac{i}{2}(\bar{x}x+\bar{y}y)_{S_I}. \tag{2.17}$$

with the Grassmanian variables x and y denoting m_{np} and s_{np} or t_{np} and r_{np}, respectively.

From the action (2.17) the following field equations may be derived:

$$i\dot{x}+\nu x-m\bar{y}=0, \tag{2.18}$$
$$i\dot{\bar{x}}-\nu\bar{x}+my=0, \tag{2.19}$$
$$i\dot{y}+\nu y+m\bar{x}=0, \tag{2.20}$$
$$i\dot{\bar{y}}-\nu\bar{y}-mx=0, \tag{2.21}$$

where

$$\nu=\frac{n+1/2}{a}. \tag{2.22}$$

In the rest of this section we give the formula for the dimensionality of $O(4)$ irreducible representation for spinors on spheres of arbitrary even dimensionality, which we shall need in the dimensional-regularisation calculations below. For this we use the technique which for scalar, vector and tensor harmonics on S^3 has been constructed by Lifshitz and Khalatnikov [27] and for spinor harmonics on S^3 is described in [22].

Let us consider flat Euclidean four-space with the metric

$$ds^2=\delta_{\mu\nu}dx^\mu dx^\nu. \tag{2.23}$$

The metric may also be written in spherical coordinates r,χ,θ,ϕ:

$$ds^2=dr^2+r^2d\Omega_3^2, \tag{2.24}$$

where

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.25}$$

Harmonics on S^3 are then obtained by considering the following homogeneous polynomials in Cartesian coordinates,

$$r^n \rho_A(\chi, \theta, \phi) = T_{AA_1 \cdots A_{n-1}\dot{A}_1 \cdots \dot{A}_{n-1}} x^{A_1 \dot{A}_1} \cdots x^{A_{n-1}\dot{A}_{n-1}}, \quad n = 1, \ldots, \tag{2.26}$$

where

$$x^{A\dot{A}} = e_\mu^{A\dot{A}} x^\mu \tag{2.27}$$

and $T_{AA_1 \cdots A_{n-1}\dot{A}_1 \cdots \dot{A}_{n-1}}$ is a constant spinor of rank $(2n - 1)$, totally symmetric in all its indices and hence traceless (in the metric ε_{AB}). It is obvious that the number of independent harmonics is defined by the number of different choices of dotted and and undotted indices which can acquire only two values: 0 or 1. For $n - 1$ dotted indices there are n opportunities while for n undotted indices there are $(n+1)$ different choices. Thus, the dimensionality of irreducible representation for Weyl spinors harmonics on S^3 is

$$\dim(n, 4) = n(n + 1). \tag{2.28}$$

In the case of a spacetime with even dimensionality d, the dimensionality of the corresponding Clifford algebra of Dirac γ - matrices is equal to 2^d; correspondingly the number of components of Dirac spinors is $2^{d/2}$, while the number of components of Weyl spinors equals $2^{(d-2)/2}$. Thus, indices A and $\dot{A}$ in the expression for constant tensor T take $2^{(d-2)/2}$ different values. In such a way the problem of calculation of dimensionality of irreducible representation for spinor harmonics reduces to the combinatorial counting of the number of possible decomposition of the set of n elements into the union of $2^{(d-2)/2}$ subsets. One has

$$\dim(n, d) = \frac{\Gamma\left(n + 2^{(d-2)/2}\right)}{\Gamma\left(2^{(d-2)/2}\right)\Gamma(n+1)} \times \frac{\Gamma\left(n + 2^{(d-2)/2} - 1\right)}{\Gamma\left(2^{(d-2)/2}\right)\Gamma(n)}. \tag{2.29}$$

3. Quantisation of fermion fields in Hamiltonian formalism and cosmological wave function

Following the Dirac procedure for systems with constraints, one may obtain from the action (2.17) the Hamiltonian

$$H_n(x, \bar{x}, y, \bar{y}) = \nu(x\bar{x} + y\bar{y}) + m(yx + \bar{x}\bar{y}), \tag{3.1}$$

where $x, y, \bar{x}, \bar{y}$ obey the Dirac-brackets relations

$$\{x, \bar{x}\}^* = -i, \quad \{y, \bar{y}\}^* = -i. \tag{3.2}$$

The total Hamiltonian is the sum of the background Hamiltonian, depending on the inflaton scalar field and the fermionic Hamiltonian, representing the sum of Hamiltonians (3.1). The total Hamiltonian vanishes, giving in such a way the Hamiltonian constraint:

$$H_0 + H_f = 0. \tag{3.3}$$

In the quantum theory, this Hamiltonian constraint becomes the Wheeler-DeWitt equation

$$[H_0 + H_f]|\Psi\rangle = 0, \tag{3.4}$$

where $|\Psi\rangle$ is the wave function of the Universe, and the fermionic Hamiltonian is obtained from its classical counterpart by replacing the variables with operators satisfying the anticommutation relations

$$[x, \bar{x}]_+ = 1, \quad [y, \bar{y}]_+ = 1. \tag{3.5}$$

In holomorphic representation they read [28, 29]

$$\bar{x} \to \frac{\partial}{\partial x}, \bar{y} \to \frac{\partial}{\partial y}. \tag{3.6}$$

With this choice, the wave function is a function of the background variables and the unbarred variables. Then there is the usual operator-ordering ambiguity in going from the classical constraint (3.3) to the Wheeler-DeWitt equation (3.4). Following [22, 18], we shall adopt the Weyl ordering, which in this case involves the substitution

$$x\bar{x} \to \frac{1}{2}\left(x\frac{\partial}{\partial x} - \frac{\partial}{\partial x}x\right). \tag{3.7}$$

With the above choice of operator ordering, the Hamiltonian (3.1) becomes the operator

$$\hat{H}_n = -\nu + \nu\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + m\left(yx + \frac{\partial^2}{\partial x \partial y}\right). \tag{3.8}$$

It is well known [30, 31, 32, 15] that in the semiclassical approximation the wave function of the Universe can be represented as a product of the wave function of macroscopical background variables (scale factor, inflaton, etc.)[1] and an infinite number of partial wave functions describing the perturbations of microscopic – in our case fermionic – fields,

$$\Psi(t, a, \varphi | x, y) = \Psi_0(t, a, \varphi) \prod_n \psi_n(t | x, y) . \tag{3.9}$$

---

[1] We work in the framework of the reduced phase space quantisation in which the wave function explicitly depends on time.

These partial wave functions satisfy Schrödinger equations with respect to the semiclassical time parameter,

$$i\dot{\psi}_n(t|x,y) = \hat{H}_n\psi_n(t|x,y). \tag{3.10}$$

Before solving (3.10) with the Hamiltonian (3.8), one should introduce the inner product for fermionic wave functions appropriate to the holomorphic representation, defined for any pair of functions f, g of the Grassmann variables x, y [28, 29]:

$$(f, g) = \int \overline{f(x,y)} g(x,y) e^{-x\bar{x} - y\bar{y}} dx d\bar{x} dy d\bar{y}. \tag{3.11}$$

Integration over $x, y, \bar{x}, \bar{y}$ is performed according to usual rules of Berezin integration [28]

$$\int dx = 0, \quad \int x dx = 1, \quad \int d\bar{x} = 0, \quad \int \bar{x} d\bar{x} = 1, \tag{3.12}$$

and likewise for y and $\bar{y}$.

We are now in a position to look for the solution of the Schrödinger equations (3.10). Taking into account the Grassmannian nature of the variables x and y, one can easily guess that this wave function is a second-order polynomial in these variables. Knowing that the no-boundary wave function of the Universe [1] is given by a Gaussian exponential, we shall look for the solution of (3.10) in the following form:

$$\psi_n(t|x,y) = v_n + \bar{v}_n xy. \tag{3.13}$$

Substituting (3.13) and (3.8) into (3.10), one can get the following system of relations for the functions v_n and $\bar{v}_n$:

$$i\dot{v}_n + \nu v_n + m\bar{v}_n = 0, \tag{3.14}$$
$$i\dot{\bar{v}}_n - \nu \bar{v}_n + m v_n = 0. \tag{3.15}$$

These equations obviously coincide with the Dirac equations (2.21) and (2.20) for y and $\bar{x}$. Now the wave function (3.13) can be rewritten in terms of the first of these two functions, the second being expressed in terms of the first one:

$$\psi_n(t|x,y) = v_n - \frac{i\dot{v}_n + \nu v_n}{m} xy \ . \tag{3.16}$$

Eliminating $\bar{v}_n$ allows one to write a second-order equation for the basis function v_n,

$$\ddot{v}_n + (-i\dot{\nu} + m^2 + \nu^2)v_n = 0, \tag{3.17}$$

which obviously has two independent solutions. The choice of a particular solution can be done on the basis of the path-integral derivation of the no-boundary wave function which automatically picks out a particular vacuum state for quantum fields on the DeSitter background. This will be done in the next section.

9

4. Path integral, Euclidean effective action and density matrix of fermions

The definition of the no-boundary wave function of fermionic fields in the holomorphic representation of the previous section can be given by the path integral over the fermionic fields in the spacetime ball with the Euclidean spacetime metric

$$ds_E^2 = d\tau^2 + a^2(\tau)d\Omega_3^2, \quad a(\tau) = \frac{1}{H}\sin H\tau, \tag{4.1}$$

which can be obtained from the Lorentzian metric by the analytic continuation in the complex plane of time, $\tau = \pi/2H + it$. For the partial wave functions this integral reads

$$\psi_n^{\mathrm{E}}(\tau|x,y) = \int Dx\, D\bar{x}\, Dy\, D\bar{y}\, e^{-I_n[x,\bar{x},y,\bar{y}]}, \tag{4.2}$$

where $I_n[x,\bar{x},y,\bar{y}]$ is the Euclidean action following from (2.17) by Wick rotation and by omitting the initial surface boundary term which vanishes because in the Hartle-Hawking prescription this surface shrinks to a point satisfying the requirement of regularity of all fields. The integration runs over fields with fixed values of the *unbarred* fermionic variables (x,y) at the final (outer) boundary S_F labelled by the Euclidean time τ. The Gaussian integration gives the answer

$$\psi_n^{\mathrm{E}}(\tau|x,y) = \mathrm{Det}\left(-\frac{d^2}{d\tau^2} + \nu' + \nu^2 + m^2\right)e^{-I_n^{\mathrm{ext}}},$$
$$I_n^{\mathrm{ext}} = \frac{1}{2}\left[\bar{x}(\tau)x(\tau) + \bar{y}(\tau)y(\tau)\right], \tag{4.3}$$

where the prime denotes derivation with respect to Euclidean time, and I_n^{ext} is the value of the Euclidean action at the classical extremal, and the functional determinant represents the one-loop contribution of quantum fluctuations near this extremal.

The classical Euclidean equations of motion easily follow from their Lorentzian counterpart

$$-x' + \nu x - m\bar{y} = 0, \tag{4.4}$$
$$-\bar{x}' - \nu\bar{x} + my = 0, \tag{4.5}$$
$$-y' + \nu y + m\bar{x} = 0, \tag{4.6}$$
$$-\bar{y}' - \nu\bar{y} - mx = 0. \tag{4.7}$$

The solution of this system of equations in the Euclidean ball $0 < \tilde{\tau} < \tau$ with fixed values $x(\tau) = x$, $y(\tau) = y$ at the boundary,

$$x(\tilde{\tau}) = \frac{u(\tilde{\tau})}{u(\tau)}x, \quad y(\tilde{\tau}) = \frac{u(\tilde{\tau})}{u(\tau)}y,$$
$$\bar{x}(\tilde{\tau}) = \frac{u'(\tilde{\tau}) - \nu u(\tilde{\tau})}{mu(\tau)}y, \quad \bar{y}(\tilde{\tau}) = \frac{-u'(\tilde{\tau}) + \nu u(\tilde{\tau})}{mu(\tau)}x, \tag{4.8}$$

can be given in terms of one function $u(\tau)$ satisfying the Euclidean version of (3.17),

$$u'' + (\nu' + m^2 + \nu^2)u = 0, \quad u(0) = \text{reg}, \tag{4.9}$$

and the condition of regularity at $\tau = 0$. Thus, with this solution the exponential of the Euclidean wave function reads $-I_n^{\text{ext}} = xy(u'(\tau) - \nu u(\tau))/mu(\tau)$. The preexponential functional determinant can also be expressed in terms of this Euclidean basis function according to the technique of [4, 34, 35, 15]. As shown in [33, 15] this determinant, when written as a product of eigenvalues λ of the differential operator, can be identically transformed as

$$\text{Det}\Big(- \frac{d^2}{d\tau^2} + \nu' + \nu^2 + m^2 \Big) = \prod_\lambda \lambda = E(0), \tag{4.10}$$

where $E(0)$ is a particular value at $\lambda = 0$ of the left-hand side of the equation

$$E(\lambda) = 0 \tag{4.11}$$

for the eigenvalues of the operator in question. In our case we have a Dirichlet problem (zero-boundary condition at the boundary of the Euclidean ball τ) for the second order differential operator with the $\lambda = 0$ eigenfunction $u(\tau)$. Therefore, $E(0) = u(\tau)$ in the equations above, so that the needed functional determinant coincides with the basis function taken at the boundary of the Euclidean ball,

$$\text{Det}\Big(- \frac{d^2}{d\tau^2} + \nu' + \nu^2 + m^2 \Big) = u(\tau). \tag{4.12}$$

The resulting Euclidean no-boundary wavefunction of fermions reads

$$\psi_n^{\text{E}}(\tau|x,y) = u \exp\Big(\frac{u' - \nu u}{mu} xy \Big) = u + (u' - \nu u)\frac{xy}{m}, \tag{4.13}$$

and, when analytically continued to the complex plane of time $\tau = \pi/2H + it$, gives exactly the wave function (3.16) that was obtained above by directly solving the Schrödinger equation. The corresponding Lorentzian basis functions

$$v(t) = u(\pi/2H + it), \quad v^*(t) = u(\pi/2H - it) \tag{4.14}$$

turn out to be, respectively, the negative and positive frequency basis function of the DeSitter invariant vacuum of fermions [36]. This concludes the justification of the Euclidean DeSitter invariant vacuum from the no-boundary path integral for fermions. Such a justification in the tree-level approximation (disregarding the preexponential factors) was achieved for bosons in [31] and for fermions in [22]. Here it is now proven up to the one-loop approximation. Note that for the case of fermions, this one-loop contribution brings a nontrivial overall time-dependent factor into the wave function (4.13) which was not correctly considered in [22][2].

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[2]In [22] the fermionic Hamiltonian was supplied by a *complex* vacuum energy term, allegedly following from the full Wheeler-DeWitt equation. This certainly leads to nonconservation of the norm of the vacuum state differing from (3.13) by a time-dependent multiplier. Rigorous derivation of the one-loop cosmological wave function, made for bosons in [4, 6, 5], cannot lead to such violations of unitarity.

Using Grassmann integration rules (3.12) one can find the positive definite square of the norm of the wave function (3.16):

$$N_n^2 \equiv (\psi_n, \psi_n) = \frac{1}{m^2} \left(|\nu v_n + i\dot{v}_n|^2 + m^2 |v_n|^2 \right). \tag{4.15}$$

Rewritten in terms of barred and unbarred components of the basis function, $N^2 = |v_n(t)|^2 + |\bar{v}_n(t)|^2$, $\bar{v}_n(t) \equiv \bar{u}_n(\pi/2H + it)$, this norm obviously features the conservation in time as a consequence of the Dirac equations of motion in Lorentzian spacetime.

Let us now go over to the reduced density matrix of the Universe induced by fermions. Integration over fermionic variables as environmental degrees of freedom leads for the full cosmological wave function (3.9) to the expression

$$\rho(a, \varphi | a', \varphi') = \Psi_0 \Psi_0'^* \left(\prod_n N_n \right) \left(\prod_n N_n' \right) \boldsymbol{D}(a, \varphi | a', \varphi'), \tag{4.16}$$

where the nontrivial decoherence effects due to fermions are contained in the infinite products of norms and the decoherence factor

$$\boldsymbol{D}(a, \varphi | a', \varphi') = \prod_n D_n(a, \varphi | a', \varphi'), \tag{4.17}$$

$$D_n(a, \varphi | a', \varphi') = \frac{v_n v_n'^*}{m^2 N_n N_n'} \left[m^2 + \nu\nu' + \left(\frac{d}{dt} \ln v_n \right) \left(\frac{d}{dt} \ln v_n'^* \right) \right.$$
$$\left. - i\nu \left(\frac{d}{dt} \ln v_n'^* \right) + i\nu' \left(\frac{d}{dt} \ln v_n \right) \right], \tag{4.18}$$

where the primes obviously denote that the corresponding quantities are calculated on the background of primed macroscopic variables (a', φ').

The fermionic contribution to diagonal elements of the density matrix reduces to the products of the norms of the partial wave functions. Let us show that, as in the bosonic case [4, 5, 6, 15], this contribution is given by the Euclidean effective action of fermion fields calculated on the spherical DeSitter instanton:

$$\prod_n N_n = \exp\left(-\frac{1}{2} \boldsymbol{\Gamma} \right), \tag{4.19}$$

$$\exp(-\boldsymbol{\Gamma}) = \prod_n \int Dx \, D\bar{x} \, Dy \, D\bar{y} \, e^{-I_n[x, \bar{x}, y, \bar{y}]}. \tag{4.20}$$

Here $I_n[x, \bar{x}, y, \bar{y}]$ is again the Euclidean action, but now it is taken on the whole sphere of the DeSitter instanton $0 < \tau < \pi/H$ on the class of fields regular at both poles $\tau = 0$ and $\tau = \pi/H$. Due to the properties of spatial harmonics, this regularity implies that the variables simply vanish there, $(x, y)(\tau_\pm) = 0$, $\tau_\pm \equiv 0, \pi/H$; that is why both of the surface terms corresponding to these poles are also vanishing. The partial contribution to the effective action is obviously given by the functional determinant of the same operator as above,

$$\exp(-\boldsymbol{\Gamma}_n) = \mathrm{Det}\left(-\frac{d^2}{d\tau^2} + \nu' + \nu^2 + m^2 \right) = \prod_\lambda \lambda = E_n(0), \tag{4.21}$$

but now with the other boundary conditions reflecting the regularity of fields at $\tau_\pm$. Therefore, the left-hand side of the equation for the zero eigenvalue $E(0)$ is here given by a qualitatively different expression. Let us derive it and, in this way, demonstrate that it is given by $E(0) = N_n^2$ – the square of the norm of the partial vacuum wave function with the quantum number n, see (4.15).

For this purpose, introduce the full Euclidean DeSitter sphere by extending the metric (4.1) up to the second pole $\tau = \pi/H$ and smoothly continue the operator $F = -d^2/d\tau^2 + (m^2 + \nu^2 + \nu')$ to the second hemisphere by observing that $\nu(\tau) = \nu(\pi/H - \tau)$. Then introduce two sets of basis functions of this operator (solutions of the corresponding homogeneous equation (4.9))

$$u''_\pm - (\nu' + m^2 + \nu^2)u_\pm = 0, \tag{4.22}$$
$$u_+(0) = \text{reg}, \quad u_-(\pi/H) = \text{reg}. \tag{4.23}$$

One certainly identifies $u_+(\tau)$ with the Euclidean basis function $u(\tau)$ introduced above and also tries as a candidate for $u_-(\tau)$ the function $u_1(\tau) = u(\pi/H - \tau)$ used in the bosonic case in [15]. Now, however, this turns out to be impossible, because in view of the relation $\nu'(\pi/H-\tau) = -\nu'(\tau)$ this function satisfies another equation, $u''_1 - (-\nu' + m^2 + \nu^2)u_1 = 0$. The remedy is to take first the new function $u_2 = (u' - \nu u)/m^2$ which, as can easily be verified, satisfies this equation with the negative sign of ν', $u''_2 - (-\nu' + m^2 + \nu^2)u_2 = 0$, and then construct its mirror image with respect to the equatorial section of $\tau = \pi/2H$, $u_-(\tau) = u_2(\pi/H - \tau)$. Then this function will satisfy the original equation and will be, by construction, regular at the pole $\tau = \pi/H$. Thus the set of two basis functions of our operator regular at the opposite poles of the DeSitter instanton is given by

$$u_+(\tau) = u(\tau), \quad u_-(\tau) = \left.\frac{u' - \nu u}{m^2}\right|_{\tau=\pi/H-\tau}. \tag{4.24}$$

In terms of these two basis functions, the equation of the zero eigenvalue of the operator can be written as the linear dependence of two two-dimensional vectors composed of these functions and their derivatives,

$$\begin{bmatrix} u_+(\tau) \\ u'_+(\tau) \end{bmatrix} - \sigma \begin{bmatrix} u_-(\tau) \\ u'_-(\tau) \end{bmatrix} = 0. \tag{4.25}$$

The left-hand side of this equation can also be written as

$$E(0) = \left(u_- u'_+ - u_+ u'_-\right)(\tau). \tag{4.26}$$

Note that in view of the Wronskian relation for the operator in question, this quantity is independent of τ. Therefore, when calculated at the equator $\tau = \pi/2H$, on using the equation for $u(\tau)$ and its corollary $u'_-(\tau) = (1/m^2)(-mu + (u' - \nu u)\nu)(\pi/H - \tau)$, the quantity $E(0)$ reads

$$E(0) = \frac{1}{m^2}\left[m^2 u^2 + (u' - \nu u)^2\right]_{\tau=\pi/2H}. \tag{4.27}$$

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In view of the analytic continuation to the Lorentzian time $\tau = \pi/2H + it$, (4.14) at $\tau = \pi/2H$ – the point of "nucleation" of the Lorentzian spacetime from the Euclidean hemisphere – we have $u(\pi/2H) = v(0) = v^*(0)$, $u'(\pi/2H) = -i\dot{v}(0) = i\dot{v}^*(0)$ and

$$E(0) = N_n^2 \tag{4.28}$$

with the square of the norm given by the expression (4.15) – the quantity which is in its turn time-independent in view of the Dirac equation for basis functions in Lorentzian spacetime.

This finally proves the relations (4.19)-(4.20) and brings the fermionic contribution to the reduced density matrix in quantum cosmology to the form

$$\rho_{\text{fermion}}(a, \varphi | a' \varphi') = \exp\left(-\frac{1}{2}\boldsymbol{\Gamma} - \frac{1}{2}\boldsymbol{\Gamma}'\right) \boldsymbol{D}(a, \varphi | a', \varphi'), \tag{4.29}$$

analogous to the bosonic contribution obtained in our previous paper [15].

This concludes the proof of the Euclidean effective-action algorithm for the diagonal element of the density matrix – the distribution function of inflationary cosmological models – derived in the bosonic case in the series of publications [4, 5, 6], and now extended to fermions. For the off-diagonal elements the effective action factors $\exp(-\frac{1}{2}\boldsymbol{\Gamma})$ and $\exp(-\frac{1}{2}\boldsymbol{\Gamma}')$ also modify the usual expression [18], but these factors are constant in t and, therefore, become negligible for late Lorentzian times, provided the decoherence effects due to $\boldsymbol{D}(a, \varphi | a', \varphi')$ rapidly grow with time. Therefore, in what follows we shall concentrate on the time evolution of the decoherence factor $\boldsymbol{D}(a, \varphi | a', \varphi')$.

5. Decoherence due to massless versus massive fermions

The structure of the decoherence factor for fermions is much more complicated than the corresponding one for bosons (cf. [15]). Their quantum state (3.16) generated according to the no-boundary prescription can be interpreted as the vacuum state corresponding to the following creation and annihilation operators:

$$a_n = \frac{|\nu v_n - i\dot{v}_n|}{\sqrt{|\nu v_n - i\dot{v}_n|^2 + m^2 |v_n|^2}} \frac{\partial}{\partial y} + \frac{|v_n| m}{\sqrt{|\nu v_n - i\dot{v}_n|^2 + m^2 |v_n|^2}} x,$$

$$a_n^+ = \frac{|\nu v_n - i\dot{v}_n|}{\sqrt{|\nu v_n - i\dot{v}_n|^2 + m^2 |v_n|^2}} y + \frac{|v_n| m}{\sqrt{|\nu v_n - i\dot{v}_n|^2 + m^2 |v_n|^2}} \frac{\partial}{\partial x},$$

$$b_n = \frac{|\nu v_n - i\dot{v}_n|}{\sqrt{|\nu v_n - i\dot{v}_n|^2 + m^2 |v_n|^2}} \frac{\partial}{\partial x} - \frac{|v_n| m}{\sqrt{|\nu v_n - i\dot{v}_n|^2 + m^2 |v_n|^2}} y,$$

$$b_n^+ = \frac{|\nu v_n - i\dot{v}_n|}{\sqrt{|\nu v_n - i\dot{v}_n|^2 + m^2 |v_n|^2}} x - \frac{|v_n| m}{\sqrt{|\nu v_n - i\dot{v}_n|^2 + m^2 |v_n|^2}} \frac{\partial}{\partial y}. \tag{5.1}$$

In the *massless* limit $m = 0$, these operators have a very simple form:

$$a_n = \frac{\partial}{\partial y},$$

$$b_n = \frac{\partial}{\partial x}, \tag{5.2}$$

and the vacuum state contains only the Grassmann-independent component following from the Schrödinger equation (3.10):

$$\psi_n = \exp\left(i \int \nu(t)dt\right). \tag{5.3}$$

From the structure of the wave function (5.3) one can see that the decoherence factor does not have a real part and hence, decoherence effects are absent. This phenomenon was already described in [18]. It is not surprising because the absence of decoherence was observed not only for conformally invariant massless fermions but for all conformally invariant fields such as the electromagnetic field and conformally-coupled massless scalar field [15].

Let us turn now to the case of a *massive* spinor field. To get the Hartle-Hawking vacuum one should take the wave function (3.16), where the function v_n can be obtained by analytic continuation from the solution of the Euclidean counterpart of the second-order equation (3.17). One should choose a solution of the Euclidean equation which is regular at the Euclidean hemisphere and serves as a germ for the Lorentzian Universe [5, 6]. Such a solution can be found for example in [36] and has the following form:

$$v_n = (1 + i \sinh Ht)^{n/2+3/4}(1 - i \sinh Ht)^{-n/2+1/4}$$
$$\times {}_2F_1(1 + im/H, 1 - im/H; n + 2; (1 + i \sinh Ht)/2). \tag{5.4}$$

We are interested in the case of large masses because at the beginning of inflation these large masses can be generated by Yukawa interactions between fermions and the inflaton scalar field (see, for example [4, 6]). In this case it would be convenient to have a uniform asymptotic expansion for the solution of (3.17). Unfortunately, in contrast to Legendre or Bessel equations (cf. [37, 33]), such an expansion cannot be written down in closed form. However, one can write down the main terms of this expansions responsible for the leading structure in the decoherence factor (4.18). The leading-in-mass m part of the basis-function frequency looks like

$$\frac{d}{dt} \ln v_n \sim \frac{i}{a}\sqrt{n^2 + m^2 a^2}. \tag{5.5}$$

Substituting it into the formula for the decoherence factor D_n, one gets the following expression for the leading term of the amplitude for the total decoherence factor:

$$|\boldsymbol{D}(a, \varphi | a', \varphi')| = \prod_n |D_n| = \exp\left(-\frac{m^2(a - a')^2}{8} I\right), \tag{5.6}$$

$$I = \sum_{n=1}^{\infty} \frac{n(n + 1)}{(n + 1/2)^2}. \tag{5.7}$$

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It is interesting to notice that in the opposite limit of small masses m one can immediately use the hypergeometrical expansion (5.4), and the leading term in mass again has the structure (5.6). This expression for small mass was obtained earlier in [18].

It is clear that the expression (5.6) contains linear ultraviolet divergences. It is in sharp contrast with the case of boson fields, studied in our preceding paper [15]. Indeed, in the case of boson fields, using conformal parametrisation of fields, it was possible to avoid ultraviolet divergences (see also [16]). Now, in spite of using the conformal parametrisation, this is not the case here. Moreover, we cannot freely redefine the fermion field by multiplying it with a power of the cosmological factor a, because the measure on the corresponding Hilbert space of states (see (3.11)) then starts depending on this multiplier. With two scale factors a and a' related to the arguments of the density matrix this would mean having two different integration measures, neither of which can be used for constructing the reduced density matrix. Thus, we should try to renormalise the decoherence factor by some regularisation procedure. This will be done in the next section by means of dimensional regularisation [19, 20].

6. Renormalised fermionic decoherence factor

We would like to calculate now the renormalised expression for the decoherence factor (5.6). This calculation is reduced to the calculation of the regularised sum (5.7). If this sum is positive, then there is a decoherence effect, while a negative sign of this quantity would imply intrinsic inconsistency of the obtained density matrix which is likely to become an unbounded operator [15].

In the framework of dimensional regularisation, $\dim(n, 4) = n(n+1)$ should be substituted by $\dim(n, d)$ from (2.29), and the eigenvalue of the three-dimensional projection of the Dirac operator, $(n + 1/2)$, should be substituted by that of the $(d - 1)$ - dimensional Dirac operator, $(n + (d - 3)/2)$. Thus, one has

$$I = \sum_{n=1}^{\infty} \frac{\dim(n, 4)}{(n + (d - 1)/2)^2}. \tag{6.1}$$

To calculate the sum of this type, one can use the Euler-Maclaurin formula:

$$I = I_1 + I_2, \tag{6.2}$$

$$I_1 = \int_0^{\infty} dy \frac{\dim(y, d)}{(y + (d - 3)/2)^2}, \tag{6.3}$$

$$I_2 = i \int_0^{\infty} \frac{dy}{\exp(2\pi y) - 1} \left(\frac{\dim(iy, d)}{(iy + (d - 3)/2)^2} - \frac{\dim(-iy, d)}{(-iy + (d - 3)/2)^2} \right). \tag{6.4}$$

The last integral (see (6.4)) is convergent at $d = 4$ and equals [38]:

$$I_2 = \frac{1}{2} \int_0^{\infty} \frac{dy \, y}{(\exp(2\pi y) - 1)(y^2 + 1/4)^2} = \frac{\pi^2}{8} - 1. \tag{6.5}$$

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The integral I_1 in (6.3) is linearly divergent and for its regularisation it is necessary to use the analytic-continuation procedure and integration by parts typical for dimensional regularisation [20, 15]. Now it is convenient to make a change of variables from y to $x = 1/y$ and to introduce the new quantity

$$f(x, d) = \frac{4\dim(1/x, d)}{[x(d-3)+2]^2} x^{2^{d/2}-2},$$ (6.6)

which is finite at $x \to \infty$. Now, the integral I_1 can be rewritten as follows:

$$I_1 = \int_0^\infty dx f(x, d) x^{-(2^{d/2}-2)}.$$ (6.7)

Integrating two times by part and discarding the terms corresponding to power-law divergences, one gets

$$I_1 = \frac{1}{(2^{d/2}-3)(2^{d/2}-4)} \int_0^\infty dx f''(x, d) x^{-(2^{d/2}-4)}.$$ (6.8)

Then, expanding expression (6.9) in powers of $(d-4)$ one has the following regularised expression:

$$I_1 = -\int_0^\infty dx \ln x f''(x, 4) - \frac{f'(0, 4)}{2^{d/2}-4} + f'(0, 4) - \frac{f'_{,d}(0, 4)}{2 \ln 2},$$ (6.9)

where $,d$ denotes the derivative with respect to the dimensionality of spacetime. The direct calculation shows that

$$f'(0, 4) = 0,$$ (6.10)

which corresponds to the absence of logarithmic ultraviolet divergences. Then

$$\int_0^\infty dx \ln x f''(x, 4) = +\frac{1}{2},$$ (6.11)

and the mixed derivative is given by

$$f'_{,d}(0, 4) = -1 + 2 \ln 2.$$ (6.12)

Substituting (6.10)–(6.12) into (6.9), one has

$$I_1 = \frac{1 - 3 \ln 2}{2},$$ (6.13)

and combining (6.13) with (6.5) one finally has

$$I = \frac{1 - 5 \ln 2}{2 \ln 2} + \frac{\pi^2}{8} < 0.$$ (6.14)

Thus, we have found a negative result which would be in strong contradiction with basic properties of a density matrix, since $\mathrm{Tr}\rho^2$ would diverge. The next section will show how the divergences can be remedied without leading to inconsistencies.

7. Bogoliubov transformation for fermions and decoherence

We have already seen in Sec. 5 that ultraviolet divergences arise in the decoherence factor of the reduced density matrix coming from fermionic perturbations, and in Sec. 6 we have obtained a dimensionally-renormalised expression which manifested wrong, anti-decoherent behaviour, breaking such fundamental properties of the density matrix as the normalisabity of its square which is equivalent to its positive-definiteness and boundedness. It happens in spite of working from the beginning in the conformal parametrisation for the spinor field, which was efficient in eliminating ultraviolet divergences in decoherence factors induced by bosonic perturbations [16, 15]. Moreover, we do not have the possibility to change the parametrisation of fermion fields by simply multiplying them by some power of the scale factor a, as it was done for bosons [15]. This happens because, as was mentioned in Sec. 5, the inner product of wave functions depending on fermion fields depends on these fields.

However, we still have some additional freedom of redefining the environmental variables by performing some kind of Bogoliubov transformations. This will lead to well-defined expressions. At the end of this section it will be shown that this transformation is analogous to the well-known Foldy-Wouthuysen transformation [21].

Thus, let us make the following Bogoliubov transformation

$$x = \alpha\tilde{x} + \beta\bar{\tilde{y}},$$
$$y = \gamma\tilde{y} + \delta\bar{\tilde{x}},$$
$$\bar{x} = \alpha^*\bar{\tilde{x}} + \beta^*\tilde{y},$$
$$\bar{y} = \gamma^*\bar{\tilde{y}} + \delta^*\tilde{x}. \tag{7.1}$$

Substituting the expressions (7.1) into the action (2.17) and requiring the conservation of the form of the kinetic term $\bar{x}\dot{x} + \bar{y}\dot{y}$, one has the following requirements for the Bogoliubov coefficients $\alpha, \beta, \gamma, \delta$:

$$\alpha^*\beta + \gamma^*\delta = 0,$$
$$|\alpha|^2 + |\beta|^2 = 1,$$
$$|\gamma|^2 + |\delta|^2 = 1. \tag{7.2}$$

It is convenient to choose the Bogoliubov coefficients in the form:

$$\alpha = \gamma,$$
$$\beta = -\delta. \tag{7.3}$$

Now, substituting the transformed Grassmann variables (7.1) into the other terms of the action (2.17), taking into account (7.2) and (7.3) and arriving at the Hamiltonian formalism, one can get the new Hamiltonian

$$\tilde{H}_n(\tilde{x}, \bar{\tilde{x}}, \tilde{y}, \bar{\tilde{y}}) = \tilde{\nu}(\tilde{x}\bar{\tilde{x}} + \tilde{y}\bar{\tilde{y}}) + \tilde{m}^*\tilde{y}\tilde{x} + \tilde{m}\bar{\tilde{x}}\bar{\tilde{y}}, \tag{7.4}$$

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where

$$\tilde{\nu} = i(\dot{\alpha}\alpha^* + \beta\dot{\beta}^*) + \nu(|\alpha|^2 - |\beta|^2) - m(\alpha\beta + \alpha^*\beta^*), \quad \tilde{\nu}^* = \tilde{\nu} \tag{7.5}$$

$$\tilde{m} = m(\alpha^{*2} - \beta^2) - 2\nu\alpha^*\beta + i(\beta\dot{\alpha}^* - \dot{\beta}\alpha^*). \tag{7.6}$$

Thus, we have found a new Hamiltonian which has the same form as the old one (3.1), with the only difference that ν and m are replaced by new functions $\tilde{\nu}$ and $\tilde{m}$ depending on Bogoliubov coefficients. Besides, the "mass" $\tilde{m}$ is complex now ($\tilde{\nu}$ remains real as one can check by using the relation (7.2) for Bogoliubov coefficients).

The Dirac equations now read

$$i\dot{\tilde{x}} + \tilde{\nu}\tilde{x} - \tilde{m}\bar{\tilde{y}} = 0, \tag{7.7}$$

$$i\dot{\bar{\tilde{x}}} - \tilde{\nu}\bar{\tilde{x}} + \tilde{m}^*\tilde{y} = 0, \tag{7.8}$$

$$i\dot{\tilde{y}} + \tilde{\nu}\tilde{y} + \tilde{m}\bar{\tilde{x}} = 0, \tag{7.9}$$

$$i\dot{\bar{\tilde{y}}} - \tilde{\nu}\bar{\tilde{y}} - \tilde{m}^*\tilde{x} = 0. \tag{7.10}$$

For generic Bogoliubov coefficients (α_n, β_n) depending on the quantum number n, the decoherence factor (4.18) becomes

$$\boldsymbol{D}(a, \varphi | a'\varphi') = \prod_n D_n(a, \varphi | a', \varphi'), \tag{7.11}$$

$$D_n(a, \varphi | a'\varphi') = \frac{v_n v_n'^*}{\tilde{m}_n \tilde{m}'^*_n N_n N_n'} \left[\tilde{m}_n \tilde{m}_n^* + \tilde{\nu}\tilde{\nu}' + \left(\frac{d}{dt} \ln v_n \right) \left(\frac{d}{dt} \ln v_n'^* \right) \right.$$
$$\left. - i\tilde{\nu} \left(\frac{d}{dt} \ln v_n'^* \right) + i\tilde{\nu}' \left(\frac{d}{dt} \ln v_n \right) \right], \tag{7.12}$$

with the new mass parameter $\tilde{m}_n$ also depending on n (ν was n-dependent even in the old parametrisation), while the approximate expression for a potentially divergent part of this factor (5.6) becomes

$$|\boldsymbol{D}(a, \varphi | a', \varphi')| = \prod_n |D_n| = \exp\left(-\frac{(a - a')^2}{8} \sum_{n=1}^{\infty} \frac{n(n+1)\tilde{m}_n \tilde{m}'^*_n}{(n + 1/2)^2} \right), \quad (7.13)$$

replacing (5.6)-(5.7).

The dependence of the new functions $\tilde{\nu}$ and $\tilde{m}$ on Bogoliubov coefficients gives new opportunities regarding the finiteness and consistency of the decoherence factor. In particular, the modified mass $\tilde{m}$ can be demanded to vanish. This would, however, lead to an absence of decoherence. For this reason, a physical principles must be invoked to select an appropriate Bogoliubov transformation: We demand that there be no effect for a *stationary* spacetime, in accordance with the idea [15] that decoherence is connected with particle creation. The decoherence factor should thus contain terms that vanish when $\dot{a}$ vanishes. Since one the main features of decoherence is its *irreversibility* [8] – and the escape of created particles is, similar to usual scattering situations, an irreversible process – we think that the above principle captures the physical effect of decoherence.

We therefore choose real α and β satisfying the equation (this is the $(\dot{\alpha}, \dot{\beta})$-independent part of $\tilde{m}$)

$$m(\alpha_n^2 - \beta_n^2) - 2\nu\alpha_n\beta_n = 0. \tag{7.14}$$

One finds the following solutions:

$$\alpha_n = \left(\frac{\sqrt{m^2 + \nu^2} + \nu}{2\sqrt{m^2 + \nu^2}} \right)^{1/2}, \tag{7.15}$$

$$\beta_n = \left(\frac{\sqrt{m^2 + \nu^2} - \nu}{2\sqrt{m^2 + \nu^2}} \right)^{1/2}. \tag{7.16}$$

If ν is independent of time (a stationary Universe), this choice of Bogoliubov coefficients results in a vanishing $\tilde{m}$; however, for a nonstationary Universe there is an imaginary contribution proportional to time derivatives of Bogoliubov coefficients,

$$\dot{\alpha}_n = \frac{m^2\dot{\nu}}{4(m^2 + \nu^2)^{3/2}} \left(\frac{\sqrt{m^2 + \nu^2} + \nu}{2\sqrt{m^2 + \nu^2}} \right)^{-1/2}, \tag{7.17}$$

$$\dot{\beta}_n = -\frac{m^2\dot{\nu}}{4(m^2 + \nu^2)^{3/2}} \left(\frac{\sqrt{m^2 + \nu^2} - \nu}{2\sqrt{m^2 + \nu^2}} \right)^{-1/2}. \tag{7.18}$$

Therefore,

$$\tilde{m}_n = i\frac{m\dot{\nu}}{2(m^2 + \nu^2)}, \tag{7.19}$$

and at large values of n this mass parameter behaves as

$$\tilde{m}_n \sim \frac{1}{n}. \tag{7.20}$$

Then, turning to the expression (7.13) one can see that the dependence of $\tilde{m}$ on n presented in (7.19) provides the suppression of ultraviolet divergences in the modified expression (7.13). The terms in the sum in the exponent of (7.13) approach for large n the expression

$$\frac{n(n+1)\tilde{m}_n\tilde{m}_n^*}{(n+1/2)^2} \rightarrow \frac{m^2\dot{a}^2}{4n^2}, \tag{7.21}$$

and the sum is explicitly convergent.

We want to conclude this section by comparing the above Bogoliubov transformations with their counterparts in a stationary flat spacetime. As it was already mentioned for the case of stationary background, coefficients α and β from Eqs. (7.16)-(7.16) eliminate mass exactly. The form of these coefficients reminds that

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of a Foldy-Wouthuysen transformation [21], but really their role is just the opposite. To explain it, let us remember that in the canonical basis (cf. with (2.9) in Sec. 2), a Dirac spinor is represented in the form

$$\Psi_C = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \qquad (7.22)$$

where two-component spinors satisfy the following Dirac equations:

$$i\frac{\partial \phi}{\partial t} = \vec{\sigma}\vec{p}\chi + m\phi,$$
$$i\frac{\partial \chi}{\partial t} = \vec{\sigma}\vec{p}\phi - m\chi. \qquad (7.23)$$

To disentangle these equations and to get rid of the so-called odd terms proportional to $\vec{\sigma}\vec{p}$, one can use a Foldy-Wouthuysen transformation

$$\Psi_C = e^{-S}\Psi_{FW}, \qquad (7.24)$$

where

$$S = -i\beta\frac{\vec{\alpha}\vec{p}}{|\vec{p}|}\theta,$$
$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix},$$
$$\theta = \frac{1}{2}\arcsin\frac{|\vec{p}|}{\sqrt{m^2 + \vec{p}^2}}. \qquad (7.25)$$

This transformation also digonalises the Hamiltonian which is now represented by a direct sum of two non-local Hamiltonians $\pm\sqrt{m^2 + \vec{p}^2}$:

$$H = \beta\sqrt{m^2 + \vec{p}^2}. \qquad (7.26)$$

It is clear that these square roots cannot be represented in configuration space by a finite set of differential operators. It is obvious also that our modified Hamiltonians (7.4) are also combined into a non-local Hamiltonian because all parameters $\tilde{\nu}, \tilde{m}$ of these partial Hamiltonians depend on the wave number n.

However, as a matter of fact, we are working in the Weyl representation for Dirac spinors (2.8) and γ matrices (2.7). In this representation, especially, massive terms are mixing in the Dirac equations (2.12) and our task is to get rid of them. This is exactly the problem that was studied in this section in two-component formalism; it can thus be called "anti-Foldy-Wouthuysen" transformation and can be in a traditional (four-spinor) formalism presented by the following operators:

$$\Psi = e^T\tilde{\Psi}, \qquad (7.27)$$

where

$$T = \tilde{\theta}\frac{\vec{\alpha}\vec{p}}{|\vec{p}|}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$
$$\tilde{\theta} = -\frac{1}{2}\arcsin\frac{m}{\sqrt{m^2 + \vec{p}^2}}. \qquad (7.28)$$

Correspondingly, the Hamiltonian is turned by this transformation into

$$H = -\beta \frac{\vec{\alpha}\vec{p}\sqrt{m^2 + \vec{p}^2}}{|\vec{p}|} \tag{7.29}$$

and is again apparently non-local.

8. Conclusions

Let us briefly recapitulate the main results of the present paper. We developed the formalism of the one-loop no-boundary wave function of the Universe in a cosmological model with fermions establishing the connection between the path integral, effective action and semiclasssical Schrödinger equation. It is worth adding here that these results are valid for the tunneling wave function [3] as well due to the fact that the structure of the microscopic perturbation part of these two wave functions is the same [39, 40]. We studied the expression for the reduced density matrix and showed that the amplitude of its decoherence factor reveals ultraviolet divergences, which was previously observed in [18]. The renormalised expression for this decoherence factor which was obtained by dimensional regularization has a wrong sign and violates the properties of density matrix as a bounded operator. However, by making special Bogoliubov transformation for fermionic variables one can get a finite expression for this decoherence factor.

Thus, as in the case of bosonic variables [15], the definition of the degrees of freedom constituting an environment is of crucial importance for the construction of consistent expressions for the reduced density matrix (see also [16, 17]). However, there is an essential difference between the fermionic and bosonic cases. In the case of bosonic degrees of freedom we have a well-defined criterion for the choice of the demarcation line between the system under consideration and the environment. Unfortunately, the situation is not so clear for fermionic degrees of freedom. We have put forward the suggestion to fix the Bogoliubov transformation by demanding that the corresponding decoherence effect would be absent if the spacetime were stationary.

One problem is that any choice of this transformation breaks locality of a fermion field; this can create difficulties at the consideration of fermions intereacting with other fields. However, in contrast with bosons when the logarithm of a decoherence factor is proportional to m^3 in the limit of large mass, for fermions that is proportional only to m^2. This fact gives support to the conclusion that bosons are much more efficient than fermions for the quantum-to-classical transition [18].

In summary, we have extended the Euclidean effective action algorithm for the quantum distribution function of cosmological models and their density matrix to fermions; this was previously only known for bosons [4, 5, 6, 40]. Together with [15] this gives a complete account of the quantum-to-classical transition at the onset of inflation – the beginning of our classical Universe.

Acknowledgements

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